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QUANTUM DIFFEOMORPHISMS AND CONFORMAL SYMMETRY

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Abstract

We analyze the constraints of general coordinate invariance for quantum theories possessing conformal symmetry in four dimensions. The character of these constraints simplifies enormously on the Einstein universe $R \times S^3$. The $SO(4, 2)$ global conformal symmetry algebra of this space determines uniquely a finite shift in the Hamiltonian constraint from its classical value. In other words, the global Wheeler-De Witt equation is *modified* at the quantum level in a well-defined way in this case. We argue that the higher moments of T^{00} should not be imposed on the physical states *a priori* either, but only the weaker condition $\langle \dot{T}^{00} \rangle = 0$. We present an explicit example of the quantization and diffeomorphism constraints on $R \times S^3$ for a free conformal scalar field.

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1. Introduction

The equality of gravitational and inertial mass led Einstein to the unshakable conviction that any theory of gravitation must respect the Equivalence Principle. General coordinate invariance is the most elegant mathematical expression of the Equivalence Principle, and consequently it was to play a central role in the development of classical general relativity. In view of the importance of general coordinate invariance in the classical theory, and in the absence of any evidence that such a pivotal guiding principle should be abandoned *a priori*, it is natural to regard diffeomorphism invariance as an essential component of any proposal for a quantum theory of gravity.

Unfortunately, although the Einstein theory is certainly coordinate invariant classically, the status of this important invariance becomes obscured in the traditional approaches to quantization. If we look beyond the myriad of technical difficulties in these approaches such as ordering ambiguities, choice of gauge, and the ill-defined nature of operators at the same spacetime point, the basic problem at their root is that there is no unique extension of the algebra of diffeomorphisms off-shell. As a result, the invariance or non-invariance of the theory becomes intertwined with the time evolution dynamics in a completely non-trivial way. Since it is virtually certain that the Einstein theory is only an effective theory which must be modified in order to construct a consistent quantum generalization, it is a great handicap to a wider exploration of possible quantum extensions to embed coordinate invariance so inextricably into the context of classical general relativity. Without a clear expression of the invariance principle in a mathematically consistent framework independent of dynamical assumptions, one cannot even “get off the ground” in quantum gravity.

It is largely for this reason of formulating diffeomorphism invariance independently of detailed dynamics that the functional integral approach to quantization offers distinct advantages [1]. There one can see a clean separation of the problem into first deter-

mining the invariant integration measure on the space of metrics and then the action functional to be integrated. It is from the invariant path integral measure that one may extract most directly the important consequences of general coordinate invariance in the quantum theory. In particular, the introduction of the conformal parameterization of the metric,

$$g_{ab}(x) = e^{2\sigma(x)} \bar{g}_{ab}(x), \quad (1.1)$$

leads quickly to the general form of the trace anomaly for matter fields in a curved background, and the effective action it induces for the σ part of the metric [2]. The Faddeev-Popov ghost determinant arises from the change of integration variables represented by (1.1), and it takes a form analogous to that in two dimensional quantum gravity (*i.e.* non-critical string theory) [1-3]. The conformal sector of quantum gravity in four dimensions may be studied then by methods closely paralleling those in two dimensions, where a great deal of progress has been made in the last decade. One consequence of this approach (in both two and four dimensions) is that although conformal symmetry is broken by the trace anomaly in a fixed background, conformal invariance is recovered when the quantum σ theory is considered. In fact, the quantum σ theory describes a conformal fixed point of quantum gravity where scale invariance is restored and the total trace anomaly of matter plus σ plus ghosts vanishes [3].

These results strongly suggest that scale invariance should play an important role in the full quantum theory. Global scale transformations are in any case just particular coordinate transformations of any spacetime that is conformally flat, of which the Friedman-Robertson-Walker metric of classical cosmology is a prime example. Hence conformal invariance should play a important role in the description of quantum gravitational effects in our Universe as well.

Motivated by these considerations we study in this paper the constraints of coordinate invariance in theories possessing global conformal symmetry. Though powerful

and elegant, the functional integral approach leaves unanswered the question of the algebra of diffeomorphisms off-shell, the Hilbert space of states, and the physical states of any theory invariant under diffeomorphisms. Addressing these issues requires a canonical operator approach to quantization. By requiring that the trace of the total energy-momentum tensor vanishes and quantizing in the conformally flat Einstein universe $R \times S^3$, we shall show that the constraints of diffeomorphism invariance vastly simplify and a detailed analysis of the constraints in four dimensions suddenly becomes possible. Otherwise we do not need to specify the Lagrangian nor supply any other dynamical information of the theory. Hence our framework and the results are quite independent of the Einstein theory (although it too is conformally invariant *on-shell* in the trivial sense that $T_a^a = 0$ follows by tracing the equations of motion).

The technical advantage of quantizing on the conformally flat spacetime $R \times S^3$ is that it admits a complete classification of states, labeled by the integers, according to their weight under conformal transformations. Hence, the constraint conditions may be solved level by level in a straightforward algebraic manner. On $R \times S^3$ rigid conformal transformations are contained explicitly in the set of volume non-preserving reparametrizations of the spatial S^3 . Hence global conformal invariance is imposed as a direct consequence of diffeomorphism invariance, and conformal transformations are treated together with and on the same footing as any other coordinate reparametrization under which the physical spectrum must be invariant. Moreover, the global conformal algebra $SO(4, 2)$ on $R \times S^3$ contains the quantum Hamiltonian as one of its generators, and the conformal weights correspond to the eigenvalues of the Hamiltonian. Because $R \times S^3$ is a product space, the time translations generated by the Hamiltonian are not mixed with spatial reparametrizations and there are no operator ordering ambiguities. This is an important feature of the approach since it allows us to determine the quantum Hamiltonian constraint uniquely and in a completely

unambiguous manner.

In $D = 2$ when quantizing the Liouville theory on $R \times S^1$, the L_0 condition on the physical states is modified by a finite c-number shift, which arises from the bc ghost action needed for quantization in covariant gauges such as (1.1) [4]. We show by explicit construction that a similar well-defined shift in the Hamiltonian condition on the physical states occurs in $D = 4$. The source of this shift lies again in the gauge fixing procedure which results in a modification of the classical Lie algebra of the finite dimensional global conformal group $SO(4, 2)$ at the quantum level in the ghost sector, *i.e* it arises from the covariant integration measure of the path integral for quantum gravity.

This modification of the classical Hamiltonian constraint by the quantum measure is important because it demonstrates explicitly that the classical Hamiltonian constraint *cannot* be maintained at the quantum level in general. Hence all attempts to implement the classical constraints of general coordinate invariance without attention to the algebra of diffeomorphisms at the quantum level must be viewed as questionable. In particular, in the canonical approach on $R \times S^3$ one does not enforce the strong form of the classical time reparametrization constraint, $T^{00} = 0$. Instead only the considerably weaker condition $\langle \dot{T}^{00} \rangle = 0$ should be imposed on matrix elements between physical states, in analogy with the procedure in $D = 2$.

That strong vanishing of T^{00} cannot be imposed as an operator condition is clear enough, since it must have non-zero commutators in order to generate time reparametrizations in the quantum theory. That only the positive frequency component of T^{00} should be required to vanish on the physical (ket) states is a consequence of the Dirac-Fock approach to the quantization of theories with constraints that we follow. Then the vanishing of the matrix elements of \dot{T}^{00} follows immediately from the fact that the negative frequency part $T^{00(-)}$ (being the Hermitian conjugate of $T^{00(+)}$)

annihilates the physical (bra) states acting to the left. However, there is *no* such condition on the zero frequency part which remains unfixed by this procedure, except for its spatial integral which is the Hamiltonian. Any condition on the full T^{00} should emerge from the *dynamics* of the theory under time evolution, and not be imposed as a kinematical constraint in the canonical approach.

The outline of the paper is as follows. In Section 2 we define the problem of the diffeomorphism constraints and classical Lie algebra of the global conformal group on $R \times S^3$, introducing necessary notation and properties of the spherical harmonic functions on S^3 which we require in the subsequent analysis. In Section 3 we discuss the modification of the classical algebra arising from the quantum measure, and construct the finite dimensional bc ghost action whose quantization leads to the c-number shift in the quantum Hamiltonian constraint. In section 4 we derive detailed expressions for the moments of the T^{0i} constraints on S^3 for a massless, conformally coupled scalar field, illustrating the algebra of spatial diffeomorphisms at the quantum level in a specific example. We show that the surviving physical state of the scalar has a correspondence with an operator of well-defined scaling dimension in four dimensions. Some properties of the harmonics on S^3 needed in the analysis are given in the Appendix for completeness.

This paper is paper I of two. In paper II we apply the general method of this paper to the analysis of the diffeomorphism constraints of the effective theory of the conformal factor generated by the trace anomaly.

2. Diffeomorphism Constraints on $R \times S^3$

We follow the parallel of radial quantization in two dimensional conformal field theories and choose as our background spacetime the static Einstein universe $R \times S^3$ with metric,

$$\bar{g}_{ab} dx^a dx^b = -dt^2 + a^2 d\Omega^2 , \quad (2.1)$$

where $d\Omega^2$ is the standard round metric on the unit three sphere. In what follows we shall set the radius, of S^3 equal to unity, $a = 1$ and drop the overbar on the metric background where it causes no confusion to do so. This metric background is conformally related to ordinary Lorentzian flat spacetime in such a way that time translations in the static Einstein space correspond to global dilations of flat Minkowski space. This is most easily demonstrated by first continuing to Euclidean signature $t \rightarrow -i\tau$:

$$d\tau^2 + d\Omega^2 = e^{2\tau}(dr^2 + r^2 d\Omega^2) = e^{2\tau} ds_{flat}^2 , \quad (2.2)$$

with

$$r = e^{-\tau} . \quad (2.3)$$

This is very convenient for our purposes since it implies that the spectrum and eigenstates of the Hamiltonian operator on $R \times S^3$ are in one-to-one correspondence with the spectrum of scaling dimensions in the conformal field theory description. This is analogous to two dimensional gravity where canonical quantization was carried out on the cylinder $R \times S^1$ [5].

The diffeomorphisms are generated by the energy-momentum tensor T^{ab} . Given any coordinate invariant action the purely spatial diffeomorphism on the sphere ξ_i is generated by the T^{0i} components of the energy-momentum tensor via

$$X_\xi \equiv - \int_{S^3} d\Omega \xi_i T^{0i} . \quad (2.4)$$

The commutator of two such spatial diffeomorphisms on S^3 is also a spatial diffeomorphism, *i.e.*

$$[iX_\xi, iX_\zeta] = iX_{[\xi, \zeta]} , \quad (2.5)$$

where $[\xi, \zeta]$ is the classical Lie bracket,

$$[\xi, \zeta]^a = \zeta^b \nabla_b \xi^a - \xi^b \nabla_b \zeta^a , \quad (2.6)$$

in the coordinate representation. This relation must remain true at the quantum level in order for the algebra of spatial diffeomorphisms to remain free of anomalies, a property we shall check *a posteriori*. The fact that the algebra of purely spatial diffeomorphisms closes upon itself is essentially a statement of kinematics in that it must be true independently of the time evolution or detailed dynamics of the theory.

Spatial diffeomorphisms may be divided into two classes: those which preserve the volume and those which do not preserve it. The volume preserving diffeomorphisms are generated by divergence-free transverse vectors, $\nabla_i \xi^i = 0$ and form a subalgebra of all spatial diffeomorphisms. On the other hand the volume non-preserving ones are generated by vectors which are pure gradients, $\xi_i = \nabla_i \phi$, and do not form a closed subalgebra since their commutators yields through their Lie bracket also the volume preserving ones.

In contrast to the T^{0i} which generate spatial diffeomorphisms, the T^{00} component generates time reparametrizations which is intertwined with dynamics, and its commutator with T^{0i} gives rise to the space-space components T^{ij} . Indeed in flat spacetime the components of the energy-momentum tensor obey the classical current algebra [6]

$$[T^{00}(t, \vec{x}), T^{0i}(t, \vec{x}')]_{flat} = -i \left(T^{ij}(\vec{x}) + T^{00}(\vec{x}') \delta^{ij} \right) \partial_j \delta^3(\vec{x} - \vec{x}') . \quad (2.7)$$

The brackets of the spatial components T^{ij} do not form a closed algebra in general. Rather the T^{ij} contain information about model dependent dynamics over and above the constraints of diffeomorphism invariance.

The lowest moment of T^{00} is simply the Hamiltonian

$$H \equiv - \int_{S^3} d\Omega \eta_a T^{0a} = \int_{S^3} d\Omega T^{00} \quad (2.8)$$

which generates rigid time translations in the product space (2.1), corresponding to the constant Killing vector

$$\eta^a = -\eta_a = (1, \vec{0}) \quad (2.9)$$

We note from the classical current algebra (2.7) that in the special case of this lowest moment of the T^{00} (and only for this moment), the Lie bracket again closes on the T^{0i} since H generates the time evolution via

$$[H, X_\xi] = -i \frac{\partial X_\xi}{\partial t} . \quad (2.10)$$

Classically demanding that the theory be invariant under both space and time coordinate transformations implies $T^{0i} = T^{00} = 0$. These constraints are automatically satisfied for any classical metric theory of gravity since the equations of motion are $T^{ab} = 0$. When we pass from the classical to the quantum theory the energy-momentum tensor becomes an operator and the Lie bracket becomes a commutator. We cannot impose the constraint of vanishing T^{0i} as an operator condition because of its non-trivial commutators (2.5) and (2.7). As in Dirac's approach to the Gauss Law constraint in electrodynamics or the quantization of two dimensional gravity, our strategy will be to demand instead that the generators X_ξ of the spatial diffeomorphism symmetry at a fixed time vanish *weakly* on the physical states. In other words, we shall require that all matrix elements of the spatial diffeomorphism generators between physical states vanish, although the operators themselves do not vanish. A sufficient condition for the weak vanishing of the matrix elements of the diffeomorphism constraints is that the *positive frequency* part $X_\xi^{(+)}$ of the generators annihilate the physical states, *i.e.*

$$X_\xi^{(+)} |phys\rangle = 0 . \quad (2.11)$$

This amounts to imposing invariance of the quantum theory under infinitesimal spatial reparametrizations. The unambiguous separation of X_ξ into positive and negative frequency components is possible only on a product space such as $R \times S^3$ possessing the timelike Killing vector (2.9). The zero frequency components must be treated separately.

Because the commutator of T^{00} does not form a closed algebra with the T^{0i} off-shell, but instead brings in other quantities which involve dynamics, we *cannot* impose the local Wheeler-De Witt condition that T^{00} (and all of its moments) vanish on the physical states [7]. Moreover, we should expect that the classical Lie bracket in (2.7) is generally modified at the quantum commutator level by anomalies. The well-known trace anomaly is the simplest example of these modifications to the classical algebra which must be taken into account in the quantum theory. Far from being kinematic in nature the exact form of such modifications to the classical algebra depends upon dynamics, and is therefore non-trivial. Hence the T^{00} and T^{ij} cannot be imposed on the physical states *a priori* without in effect knowledge of the full quantum theory right from the outset.

However, most of the classical time reparametrization invariance is recovered in the quantum theory by the imposition of the volume non-preserving spatial diffeomorphism constraints since

$$\int_{S^3} d\Omega (\nabla_i \phi) T^{0i} = \partial_t \int_{S^3} d\Omega \phi T^{00} \equiv \dot{T}_\phi , \quad (2.12)$$

for time-independent ϕ . By Fourier transforming in time it is clear that this relation together with (2.11) guarantees the weak vanishing of all the matrix elements of T^{00} containing non-zero frequencies,

$$\langle phys' | \dot{T}_\phi | phys \rangle = 0 . \quad (2.13)$$

Hence, not the Wheeler-De Witt condition of vanishing T^{00} but the weaker condition of vanishing matrix elements of \dot{T}^{00} is imposed on the physical Hilbert space of states.

The zero frequency component of T^{00} is *not* fixed by this condition. Finally, the lowest moment of the zero frequency component, the global Hamiltonian constraint of vanishing H also should not be imposed uncritically on the physical states. We shall see precisely how it should be modified by a finite subtraction in the quantum treatment of the next section.

It is instructive to compare and contrast this situation in four dimensions with two dimensions, on $R \times S^1$ [4]. In $D = 2$ the space-space component T^{11} is determined by T^{00} and the trace of the energy-momentum tensor, which is zero classically for conformally invariant matter. This means that there are no new operators in the spatial components and the full energy-momentum algebra does close. In fact, the moments of T^{00} and T^{01} are

$$\begin{aligned} T_N^{00} &= L_N e^{-iNt} + \bar{L}_{-N} e^{iNt} \\ T_N^{01} &= L_N e^{-iNt} - \bar{L}_{-N} e^{iNt} , \end{aligned} \tag{2.14}$$

where L_N and \bar{L}_{-N} are the Virasoro generators for left and right movers on the circle S^1 . Quantum mechanically there is an anomaly which appears only in the form of a c-number central extension of the classical algebra (2.7). In fact, the Virasoro algebra,

$$[L_M, L_N] = (M - N)L_{M+N} + \frac{c}{12}(M^3 - M)\delta_{M,-N}$$

for both the left and right movers implies (at $t = 0$),

$$\begin{aligned} [T_M^{01}, T_N^{01}] &= (M - N)T_{M+N}^{01} \quad \text{but} \\ [T_M^{00}, T_N^{01}] &= (M - N)T_{M+N}^{00} + \frac{c}{6}(M^3 - M)\delta_{M,-N} . \end{aligned} \tag{2.15}$$

Therefore, the purely spatial quantum algebra is identical to the classical Lie algebra (2.5) and free of any anomaly, but the second commutator (2.15) acquires a quantum correction with the central charge c proportional to the trace anomaly coefficient. The resulting algebra of both space and time reparametrizations is a local conformal algebra which has non-trivial representations in conformal field theories. This local Virasoro

algebra is also isomorphic to that obtained from the positive and negative frequency spatial moments (L_N and \bar{L}_N with N positive or negative respectively) of the T^{01} only. Moreover, the Hamiltonian operator H ($= L_0 + \bar{L}_0$ in $D = 2$) requires a finite subtraction. In $D = 4$ there is no infinite dimensional local conformal symmetry, but there is still a global conformal group of which H is one generator, which results from the separation of the moments of T^{0i} into positive and negative frequencies, as we now show.

The symmetry group of the spatial S^3 is $O(4) \cong SU(2) \times SU(2)/Z_2$, and we shall make extensive use of this symmetry group in our development. The finite dimensional representations of $SU(2) \times SU(2)$ are characterized by two angular momentum quantum numbers (J_1, J_2) with J_1 and J_2 taking on integer or half-integer values. The scalar spherical harmonics on S^3 belong to the (J, J) representation of $SU(2) \times SU(2)$. We denote these scalar harmonic functions on S^3 by $Y_{Jmm'}$ where m and m' are the magnetic quantum numbers of the two $SU(2)$ subgroups, or more compactly by Y_{JM} with $M \equiv (m, m')$ denoting the pair of $SU(2)$ magnetic indices taking $(2J+1)^2$ distinct values in total. Explicit representations for the Y_{JM} will not be needed, but are given for completeness in the Appendix in terms of standard $SU(2)$ transformation Wigner D functions, where further properties of these functions are catalogued as well [8].

Because $R \times S^3$ is conformally flat, it has fifteen conformal Killing vectors ω_a , satisfying the conformal Killing equation,

$$(L\omega)_{ab} \equiv \nabla_a \omega_b + \nabla_b \omega_a - \frac{1}{2} g_{ab} \nabla^c \omega_c = 0 , \quad (2.16)$$

the maximal number for a four dimensional manifold, and the same number as flat Minkowski spacetime. Corresponding to each conformal Killing vector there is a Noether charge,

$$\int_{S^3} d\Omega \omega_a T^{0a}$$

which is time-independent because of the conservation of the current, $\omega_b T^{ab}$ upon using (2.16) and $T_a^a = 0$.

The six purely spatial Killing vectors of S^3 (which are conformally mapped into the three rotations and three boosts of Minkowski spacetime) clearly satisfy this equation. These six solutions of (2.16) have vanishing time component and may be denoted as $\rho = (0, \rho^i)$ where the spatial index i runs from 1 to 3. It is not difficult to find an explicit representation of the ρ^i in terms of the Y_{JM} harmonic functions on S^3 :

$$\rho_{M_1 M_2}^i = i \frac{V}{4} Y_{\frac{1}{2} M_1}^* \overleftrightarrow{\nabla}^i Y_{\frac{1}{2} M_2} \quad (2.17)$$

where $V = 2\pi^2$ is the volume of the unit S^3 . This anti-symmetrized form makes it clear that

$$\nabla_i \rho_{M_1 M_2}^i = 0 \quad \text{and} \quad \left[\rho_{M_1 M_2}^i \right]^* = \rho_{M_2 M_1}^i \quad (2.18)$$

where ∇ is the covariant derivative with respect to the round metric on S^3 . In addition, because of eq. (A.6) of the Appendix, the ρ^i obey the symmetry relations,

$$\rho_{M_1 M_2}^i = -\tilde{\rho}_{M_1 M_2}^i \equiv -\epsilon_{M_1} \epsilon_{M_2} \rho_{-M_2 -M_1}^i \quad (2.19)$$

These relations together with (2.18) reduce the number of linearly independent real vectors to precisely the required six Killing vectors of S^3 .

The constant Killing vector (2.9) which generates time translations on $R \times S^3$ and is conformally mapped to the global dilation generator of flat Minkowski spacetime is clearly a solution of eq. (2.16) as well.

The remaining eight proper conformal Killing vectors, denoted by $\kappa_M^{a(\pm)}$ are easily found in terms of the spherical harmonics on S^3 in the form,

$$\kappa_M^{(+)} = -\frac{\sqrt{V}}{2} e^{it} (Y_{\frac{1}{2} M}^*, i \vec{\nabla} Y_{\frac{1}{2} M}^*) = \left[\kappa_M^{(-)} \right]^* \quad (2.20)$$

These eight $\kappa_M^{(\pm)}$ correspond in flat space to four translations plus four special conformal transformations, and satisfy

$$\nabla_a \kappa_M^{a(\pm)} = \dot{\kappa}_M^{0(\pm)} + \nabla_i \kappa_M^{i(\pm)} = \pm 4i \kappa_M^{0(\pm)} \quad (2.21)$$

Together these $6 + 1 + 8 = 15$ conformal Killing fields generate the Lie algebra of the $SO(4, 2)$ group of conformally flat 4 dimensional spacetime. Classically this may be seen at the level of the algebra of the Lie derivatives. Recall that the action of an infinitesimal coordinate transformation generated by an arbitrary vector ξ^a on a field density ϕ is

$$\delta_\xi \phi = \xi^a \nabla_a \phi + \frac{w}{4} \phi (\nabla_a \xi^a) , \quad (2.22)$$

where w is, by definition, the conformal weight of ϕ . From this definition we find that the effect of two infinitesimal transformations ξ and ζ performed in different orders is

$$[\delta_\xi, \delta_\zeta] \phi = \delta_{[\xi, \zeta]} \phi, \quad (2.23)$$

where $[\xi, \zeta]$ is the classical Lie bracket defined by (2.6).

If we now use these general properties of Lie derivatives applied to the 15 conformal Killing fields of $R \times S^3$, we find

$$\begin{aligned} [\eta, \kappa_M^{(\pm)}] &= \mp \kappa_M^{(\pm)} \\ [\kappa_{M_1}^{(\pm)}, \kappa_{M_2}^{(\mp)}] &= \pm 2\eta \delta_{M_1 M_2} + 2\rho_{M_1 M_2} \\ [\kappa_M^{(\pm)}, \rho_{M_1 M_2}] &= \delta_{M M_2} \kappa_{M_1}^{(\pm)} - \epsilon_{M_1} \epsilon_{M_2} \delta_{M - M_1} \kappa_{-M_2}^{(\pm)} \\ [\rho_{M_1 M_2}, \rho_{M_3 M_4}] &= \delta_{M_1 M_3} \rho_{M_2 M_4} + \text{permutations} , \end{aligned} \quad (2.24)$$

with all other Lie brackets vanishing or obtained from these by symmetry relations. This classical algebra of Lie brackets is isomorphic to $SO(4, 2)$ and will go over to the quantum commutator Lie algebra with the important modification of a c-number shift in the right side of the second relation in (2.24), as we shall show in the next section.

The quantum generators corresponding to these 15 conformal Killing vectors may now be constructed. The Hamiltonian has been given already by (2.8). The generators of rotations on S^3 are

$$R_{M_1 M_2} \equiv X_{\rho_{M_1 M_2}} = - \int_{S^3} d\Omega \rho_{i M_1 M_2} T^{0i} , \quad (2.25)$$

which are time-independent volume preserving diffeomorphisms. Since the physical states must be invariant under these particular volume preserving spatial diffeomorphisms, we require

$$R_{M_1 M_2} |phys\rangle = 0 . \quad (2.26)$$

The generators of the eight proper conformal Killing transformations are

$$\begin{aligned} K_M^{(\pm)} &\equiv - \int_{S^3} d\Omega \kappa_{aM}^{(\pm)} T^{0a} \\ &= - \int_{S^3} d\Omega \kappa_{0M}^{(\pm)} (T^{00} \pm i \partial_t T^{00}) \\ &= \mp 2i \int_{S^3} d\Omega \kappa_{0M}^{(\pm)} \partial_t T^{00(\pm)} \\ &= e^{\pm it} \dot{T}_{\frac{1}{2}M}^{(\pm)} \end{aligned} \quad (2.27)$$

where we have used (2.20), the conservation of T^{ab} an integration by parts, and the following definition for the moments of the time derivative of the energy density:

$$\dot{T}_{JM}^{(+)} \equiv -i\sqrt{V} \int_{S^3} d\Omega Y_{JM}^* \partial_t T^{00(+)} . \quad (2.28)$$

In passing to the third line of (2.27) we have also made use of the fact that the Noether charges $K^{(\pm)}$ are time-independent, so that the exponential time dependence in $\kappa_{0M}^{(\pm)}$ must be canceled by that in T^{00} . In other words, the $\kappa_{0M}^{(\pm)}$ act as projectors onto time dependence $\exp(\mp it)$ of $\partial_t T^{00}$ which selects precisely the positive or negative frequency moments, $\dot{T}_{\frac{1}{2}M}^{(\pm)}$ of the last line, provided the energy-momentum tensor is traceless. Hence the positive frequency component, $K_M^{(+)}$ is just proportional to $\dot{T}_{JM}^{(+)}$ for $J = \frac{1}{2}$, and is automatically enforced to vanish on the physical states from our general condition (2.11).

The above discussion shows that $8 + 6 = 14$ of the global special conformal transformations of $R \times S^3$ are identified with the lowest moments of spatial diffeomorphism generators X_ξ , once they are separated into positive, negative and zero frequency components. Together with the Hamiltonian H they form the Lie algebra of the global

conformal group $SO(4, 2)$. In two dimensions the conformal group is infinite dimensional, and this identification of diffeomorphism generators with conformal group generators is extended to all higher moments. There are an infinite number of conserved currents and charges of the conformal Virasoro algebra, and as shown in (2.14) they are in one-to-one correspondence with the generators of positive, negative and zero frequency spatial diffeomorphism generators, together with the Hamiltonian $L_0 + \bar{L}_0$. This extended Virasoro algebra is also essential to the understanding of the invariance of the theory under finite diffeomorphisms. As we see by the above discussion, this situation does not generalize to higher dimensions, where the conformal group is finite dimensional. This means, in particular that the commutators of the higher moments of the separate positive and negative frequency components of the spatial diffeomorphism generators $X_\xi^{(\pm)}$ do not necessarily form a closed algebra, or if they do, this algebra is unknown. Hence the full algebraic structure of diffeomorphism invariance in four dimensions is not determined by our approach, and remains an open problem.

3. The Quantum Hamiltonian Constraint

In this section we wish to calculate the quantum contribution to the Hamiltonian constraint on the product space $R \times S^3$. The Hamiltonian physical state condition will be of the form,

$$H|phys\rangle = a|phys\rangle$$

where a is the finite subtraction constant we wish to determine. The classical value of $a = 0$ will turn out to be untenable at the quantum level. As in string theory on the two dimensional world sheet, this subtraction may be determined by consistency with the global conformal algebra of the Fadeev-Popov ghost-antighost system. Before entering into the detailed analysis of ghosts, let us give a heuristic argument why a value different from zero should be expected, based on experience in $D = 2$.

Any physical state in a diffeomorphic invariant theory should be created by an invariant scalar, which may be expressed as the spacetime integral $\int d^4x$ of an operator with conformal weight 4 (*i.e.* a volume density), operating on the vacuum state invariant under the symmetry group of $R \times S^3$, $|0\rangle$,

$$|phys\rangle = \mathcal{O}_4|0\rangle, \tag{3.1}$$

which is annihilated by H ,

$$H|0\rangle = 0. \tag{3.2}$$

Now under the conformal transformation which maps $R \times S^3$ to flat Euclidean space (2.2) the Hamiltonian generator of time translations is mapped to the global dilation generator of flat space, which we denote by \bar{H} . In particular this means that in flat space

$$[\bar{H}, \bar{\mathcal{O}}_4] = 4\bar{\mathcal{O}}_4, \tag{3.3}$$

with $\xi^a = x^a$ and $w = 4$ in eq. (2.22). Performing the transformation of this relation back to $R \times S^3$ allows us to remove the overbars. Then operating on the invariant

vacuum and using (3.1) and (3.2) yields

$$H|phys\rangle = 4|phys\rangle . \quad (3.4)$$

We conclude on the basis of these simple dimensional considerations that $a = 4$.

This interpretation is completely consistent with the corresponding value of H in closed string theory, namely 2 on the world-sheet of 2 spacetime dimensions. At the canonical algebra level this 2 arises from the anomalous commutator in the global conformal algebra $SU_L(1,1) \times SU_R(1,1) \cong SO(2,2)$ of the ghost energy-momentum tensor, introduced in order to fix the gauge completely in path integral quantization of the string. There is no anomaly in the global algebra of matter energy-momentum tensors, so that the ghosts alone contribute to the shift of a to 2 from its classical value of 0. In order to justify the heuristic dimensional argument leading to eq. (3.4) we shall need to carry out the analogous construction of the ghost energy-momentum tensor and the global algebra of its moments in four dimensions.

Let us first recall the quantum measure in the path integral over metrics in the conformal decomposition (1.1). In several earlier papers [1-3] we introduced the natural orthogonal splitting on the space of metric deformations induced by this decomposition, namely,

$$\delta g_{ab} = \left(2\delta\sigma + \frac{1}{2}\nabla_c\xi^c\right) g_{ab} + (L\xi)_{ab} + h_{ab}^\perp, \quad (3.5)$$

where h_{ab}^\perp is a transverse, tracefree symmetric tensor, and L is defined by (2.16). Then the path integral measure on the space of all metrics may be written as a product over the three coordinates (σ, ξ, h^\perp) together with a Jacobian for the change of coordinates, given by

$$J = \left[\det' \left(L^\dagger L\right)\right]^{\frac{1}{2}}, \quad (3.6)$$

where the prime indicates the restriction to the non-zero mode subspace of the operator

L defined in (2.16). The two operators L and L^\dagger are individually conformally covariant,

$$L = e^{2\sigma} \bar{L} e^{-2\sigma} \quad , \quad L^\dagger = e^{-4\sigma} \bar{L}^\dagger e^{2\sigma} , \quad (3.7)$$

although their product $L^\dagger L$ is not. Let us remark that since L maps vectors into symmetric, traceless tensors and L^\dagger maps traceless, symmetric tensors into vectors, there is no meaning to the determinants of each of these operators individually. However, because of the one-to-one correspondence between eigenvectors of $L^\dagger L$, $\xi^{(\lambda)}$ with eigenvalue λ and eigentensors of LL^\dagger in the range of L by the relation,

$$(LL^\dagger)(L\xi^{(\lambda)}) = L(L^\dagger L)\xi^{(\lambda)} = \lambda(L\xi^{(\lambda)}) , \quad (3.8)$$

the spectrum of $L^\dagger L$ is the same as that of LL^\dagger , and the determinants of these two operators are equal when restricted to their non-zero mode subspaces. Hence the Jacobian (3.6) may be reproduced by introducing the ghost action,

$$S_{gh} = -\frac{i}{4\pi^2} \int d^4x \sqrt{-g} \, b^{\alpha\beta} (Lc)_{\alpha\beta} , \quad (3.9)$$

where all fields and operators now transform covariantly under the conformal decomposition (1.1). The integration over the anticommuting Grassman fields $b^{\alpha\beta}$ and c_α , restricted to the non-zero mode subspaces of L and L^\dagger , yields precisely the Jacobian J of eq. (3.6). The normalization factor in S_{gh} is arbitrary at this point, and is chosen in order to yield a conveniently normalized hermitian energy-momentum tensor for the ghosts below.

One complication to this ghost construction arises from the zero modes of L and L^\dagger . The zero modes of L in (2.16) are the conformal Killing vectors ω of the background corresponding to the generators of the finite dimensional global conformal group $SO(4, 2)$ in the conformally flat $R \times S^3$ metric. The zero mode space of L^\dagger is spanned by all transverse, tracefree symmetric tensors which is an infinite dimensional space

for $D = 4$. The reason for the difference with $D = 2$ is clear: there are no physical transverse graviton modes in $D = 2$ where all metrics are locally conformally flat. It is only by including the contributions of the zero mode spaces of L and L^\dagger that the σ dependence of measure may be extracted in the form of a local action $\Gamma[\bar{g}; \sigma]$ and the gravitational measure presented in the factorized form,

$$[\mathcal{D}(e^{2\sigma}\bar{g})] = e^{-\Gamma[\bar{g}; \sigma]} \left(\frac{\det' L^\dagger L}{\det < \omega_j | \omega_k >} \right)_{\bar{g}}^{1/2} [\mathcal{D} Vol(Diff_0)] [\mathcal{D}\sigma] [\mathcal{D}\bar{g}^\perp] . \quad (3.10)$$

In this form the integration over the gauge orbit of coordinate diffeomorphisms $Vol(Diff_0)$ (continuously connected to the identity) has been factorized explicitly and may be divided out. The σ dependence of the measure is expressed purely by the local action $\Gamma[\bar{g}; \sigma]$ which was obtained in refs. [2] and [3], and is studied in detail in paper II. The physical graviton modes are contained in \bar{g}^\perp which must be integrated with the measure $\mathcal{D}\bar{g}^\perp$ together with some invariant local action $S_{inv}[\bar{g}]$ in $D = 4$.

The existence of a finite number of conformal Killing vectors $N_{CKV} = 15$ means that the dimension of the range of L has decreased by N_{CKV} from the generic non-conformally flat background metric. Since the dimension of the space of metrics should not decrease discontinuously by increasing the symmetry of the background through continuous deformations, these N_{CKV} diffeomorphisms should reappear in the other parts of the general decomposition of metric deformations of (3.5). Phrased differently, when there are N_{CKV} zero modes of L the condition that the traceless part of δg_{ab} satisfy the Lorentz-Landau gauge condition

$$\nabla^b h_{ab}^\perp = 0 \quad (3.11)$$

does not fix the gauge completely, and there should be precisely N_{CKV} additional transverse, tracefree tensors h_{ab}^\perp (or scalars σ) corresponding to this residual gauge freedom. It is not difficult to construct the transverse, traceless, symmetric tensor

corresponding to each CKV of $R \times S^3$. To do so let us present the general h_{ab}^\perp in the form,

$$h_{ab}^\perp = \begin{pmatrix} u & v_i \\ v_i & \frac{u}{3}g_{ij} + h_{ij}^{TT} \end{pmatrix}, \quad (3.12)$$

where the h_{ij}^{TT} are transverse and traceless with respect to the spatial metric on S^3 , and

$$\begin{aligned} \dot{u} &= \nabla_i v^i \\ \dot{v}_i &= \frac{1}{3}\nabla_i u \end{aligned} \quad (3.13)$$

are the conditions that h_{ab} be transverse with respect to the full four-metric on $R \times S^3$. Now the key observation in the construction is to recognize the similarity between the transversality conditions (3.13) and the time-time and time-space components of the conformal Killing equations (2.16) $L\omega = 0$, namely,

$$\begin{aligned} \dot{\omega}^0 &= \frac{1}{3}\nabla_i \omega^i \\ \dot{\omega}_i &= \nabla_i \omega^0. \end{aligned} \quad (3.14)$$

By identifying

$$\begin{aligned} u &\rightarrow 3\omega^0, & \text{and} \\ v_i &\rightarrow \omega_i \end{aligned} \quad (3.15)$$

we observe that any solution of (3.14) generates a transverse, tracefree tensor (3.12). Of the infinite number of transverse, tracefree tensors satisfying (3.13) we thereby select only those satisfying also the space-space part of $L\omega = 0$, namely

$$\nabla_i \omega_j + \nabla_j \omega_i = \frac{1}{2}g_{ij}(\dot{\omega}^0 + \nabla_k \omega^k) = \frac{2}{3}g_{ij}\nabla_k \omega^k. \quad (3.16)$$

and no h_{ij}^{TT} physical graviton component.

Since $R \times S^3$ has exactly $N_{CKV} = 15$ solutions of the equations (3.14) and (3.16), the above identification constructs exactly 15 transverse, tracefree tensors corresponding to the conformal Killing vectors. The fact that these transverse tensors are in one-to-one correspondence with conformal Killing vectors and independent of the detailed dynamics of the physical graviton modes implies that they should not appear

in the expansion of any generally covariant action around a background possessing these conformal Killing symmetries. Indeed, we have checked explicitly that for the Weyl tensor squared action, when expanded to quadratic order around $R \times S^3$ these 15 tensor modes drop out of the quadratic action, *i.e.* are solutions of the differential equation obtained from the action. These 15 solutions are the only modes which are not in the ordinary physical space of solutions of the fourth order wave equation of the linearized Weyl theory.

In the case of the Einstein-Hilbert action the $R \times S^3$ geometry is not a solution of the equations of motion in the absence of matter. However, if a positive cosmological term is added to the action then de Sitter spacetime is a conformally flat solution with the global topology of $R \times S^3$, possessing exactly 15 conformal Killing vectors. In the expansion about this background one finds once again exactly 15 modes (10 in h^\perp and 5 in σ in this case) which drop out the quadratic action, and which are not contained in the ordinary h_{ij}^{TT} physical graviton solutions of the wave equation.

With this understanding of the origin of the ghost action (3.9) in the covariant path integral and the special role played by the zero modes of L we turn now to the canonical quantization procedure on the space $R \times S^3$. The ghost system must be quantized together with the σ , h^\perp and matter fields of the theory. The equations of motion for the ghost field c_α obtained by varying S_{gh} with respect to $b^{\alpha\beta}$ is

$$(Lc)_{\alpha\beta} = 0 , \quad (3.17)$$

so that c_α may be expanded in terms of the 15 conformal Killing vectors of $R \times S^3$,

$$c^\alpha(x) = \eta^\alpha c_0 + \rho_{M_1 M_2}^\alpha(x) c_{M_1 M_2} + \kappa_M^{\alpha(-)}(x) c_M + \kappa_M^{\alpha(+)}(x) c_M^\dagger . \quad (3.18)$$

The tensor ghost field $b_{\alpha\beta}$ may be expanded then in the corresponding 15 transverse,

tracefree modes constructed by relations (3.12) and (3.15) above:

$$\begin{aligned}
b_{00}(x) &= u_0 b_0 + u_M(x) b_M + u_M^*(x) b_M^\dagger \\
b_{0i}(x) &= v_{i \ M_1 M_2}(x) b_{M_1 M_2} + v_{i \ M}(x) b_M + v_{i \ M}^*(x) b_M^\dagger \\
b_{ij}(x) &= \frac{1}{3} g_{ij} \left(u_0 b_0 + u_M(x) b_M + u_M^*(x) b_M^\dagger \right) .
\end{aligned} \tag{3.19}$$

The normalization of the tensor modes may be fixed by taking

$$\begin{aligned}
u_0 &= \eta^0 = 1 \\
u_M &= 2\kappa_M^{0(-)} \\
v_M^i &= \frac{2}{3} \kappa_M^{i(-)} \\
v_{M_1 M_2}^i &= \frac{1}{2} \rho_{M_1 M_2}^{i*}
\end{aligned} \tag{3.20}$$

so that the variation of the ghost action takes the canonical form, and canonical anti-commutation relations may be imposed:

$$\begin{aligned}
\{b_0, c_0\} &= 1 \\
\{b_{M_1}, c_{M_2}^\dagger\} &= \{b_{M_1}^\dagger, c_{M_2}\} = \delta_{M_1 M_2} \\
\{b_{M_1 M_2}, c_{M_3 M_4}\} &= \delta_{M_1 M_3} \delta_{M_2 M_4} - \epsilon_{M_1} \epsilon_{M_2} \delta_{M_1 - M_4} \delta_{M_2 - M_3}
\end{aligned} \tag{3.21}$$

where ϵ_M is defined in (A.6).

The energy-momentum tensor of the ghosts obtained by varying S_{gh} is

$$T_{gh}^{\mu\nu} = \frac{i}{2\pi^2} \left(\nabla_\alpha (b^{\mu\nu} c^\alpha) + 2b_\alpha^{(\mu} \nabla^{\nu)} c^\alpha - \frac{1}{2} b^{\mu\nu} \nabla_\alpha c^\alpha \right) . \tag{3.22}$$

By substituting the mode expansions (3.18) and (3.19) into this expression and using (3.20), (2.17) and (2.20), we obtain

$$H^{gh} \equiv \int_{S^3} d\Omega : T_{gh}^{00} := \sum_M \left(b_M^\dagger c_M + c_M^\dagger b_M \right) , \tag{3.23}$$

which is the Hamiltonian of the finite dimensional ghost system in normal ordered form.

The other non-vanishing moments of the ghost energy-momentum tensor we require are

$$\begin{aligned}
K_M^{(+)\,gh} &\equiv - \int_{S^3} d\Omega \, \kappa_{a\,MM'}^{(+)} T_{gh}^{0a} \\
&= 2b_0 c_M + b_M c_0 + \sum_{M'} (2b_{M'} c_{M'M} + b_{MM'} c_{M'}) \\
K_M^{(-)\,gh} &\equiv - \int_{S^3} d\Omega \, \kappa_{a\,M}^{(-)} T_{gh}^{0a} \\
&= 2c_M^\dagger b_0 + c_0 b_M^\dagger + \sum_{M'} (2c_{MM'} b_{M'}^\dagger + c_{M'}^\dagger b_{M'M})
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
R_{M_1 M_2}^{gh} &\equiv -\frac{1}{2} \int_{S^3} d\Omega \, \rho_{i\,M_1 M_2} (T_{gh}^{0i}[b, c] - T_{gh}^{0i}[c, b]) = \\
&\sum_M (b_{M M_2} c_{M M_1} - b_{M_1 M} c_{M_2 M}) + b_{M_2}^\dagger c_{M_1} - b_{M_1} c_{M_2}^\dagger - s_{M_1} s_{M_2} (b_{-M_1}^\dagger c_{-M_2} - b_{-M_2} c_{-M_1}^\dagger) \\
&\quad - (b \leftrightarrow c) \;,
\end{aligned} \tag{3.25}$$

where this last expression has been anti-symmetrized in the ghost and anti-ghost operators.

With these expressions for the generators of the global conformal group, and making use of the properties of the first few scalar and vector harmonics catalogued in the Appendix, it is a straightforward (but tedious) exercise to compute their commutators and verify that they do close to the algebra of $SO(4, 2)$ with the important modification of a *c-number shift* of -4 in the Hamiltonian H^{gh} . This shift may be found by calculating the commutator,

$$\left[K_{M_1}^{(+)\,gh}, K_{M_2}^{(-)\,gh} \right] = 2\delta_{M_1 M_2} (H^{gh} - 4) + 2R_{M_1 M_2}^{gh} \tag{3.26}$$

The most rapid way to obtain the shift of -4 is to calculate the expectation value of the above commutator in the ghost vacuum state which is annihilated by the positive frequency destruction operators, c_M and b_M while for the zero frequency operators it may be chosen to obey (see eqs (3.21)):

$$\begin{aligned}
c_0|0\rangle &= c_{M_1 M_2}|0\rangle = 0 \quad \text{but} \\
\langle 0|b_0 &= \langle 0|b_{M_1 M_2} = 0 \;.
\end{aligned} \tag{3.27}$$

This state is invariant under the maximal subgroup of $SO(4, 2)$ generated by H^{gh} , $K_M^{(+)gh}$ and $R_{M_1 M_2}^{gh}$. Then we find

$$\begin{aligned}
\langle 0 | \left[K_{M_1}^{(+)gh}, K_{M_2}^{(-)gh} \right] | 0 \rangle &= 2 \langle 0 | b_{M_1} c_0 c_{M_2}^\dagger b_0 | 0 \rangle + 2 \sum_{M M'} \langle 0 | b_{M_1 M} c_M c_{M_2 M'} b_{M'}^\dagger | 0 \rangle \\
&= -2 \delta_{M_1 M_2} - 2 \sum_{M'} \left(\delta_{M_1 M_2} \delta_{M' M'} - \delta_{M_1 M'} \delta_{M' M_2} \right) \quad (3.28) \\
&= -8 \delta_{M_1 M_2}
\end{aligned}$$

which is equivalent to a shift of -4 in H^{gh} from (3.26).

The shift of -4 is completely quantum mechanical in origin and comes only from the bc ghost system. Matter fields exhibit no such shift, as we shall see explicitly in the example of the next section and in paper II. As a consequence, the Hamiltonian constraint which is the lowest moment of the classical Wheeler-DeWitt condition $T^{00} = 0$ is *modified by a definite c-number shift*, and obeys (3.4) on any physical state of a coordinate invariant theory with zero energy-momentum trace $T_a^a = 0$ on $R \times S^3$.

4. Quantization of a Conformal Scalar on $R \times S^3$

As a concrete realization of the general structure of the diffeomorphism generators on $R \times S^3$ we present here the results for the Fock representation in the simplest example, a conformal free scalar field with action,

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} \left(g^{ab} \nabla_a \Phi \nabla_b \Phi + \frac{1}{6} R \Phi^2 \right) . \quad (4.1)$$

On $R \times S^3$ with unit radius, the field Φ may be expanded in functions of the form $\exp(-i\omega t)Y_{JM}$, and the wave operator $\square - 1$ factorizes as

$$(\square - 1) \exp(-i\omega t)Y_{JM} = (\omega^2 - (2J+1)^2) \exp(-i\omega t)Y_{JM} , \quad (4.2)$$

so that Φ may be written in terms of creation and destruction operators

$$\Phi = \sum_{JM} \frac{1}{\sqrt{2(2J+1)}} \left(e^{-i(2J+1)t} Y_{JM} \varphi_{JM} + e^{i(2J+1)t} Y_{JM}^* \varphi_{JM}^\dagger \right) , \quad (4.3)$$

obeying canonical commutation relations

$$\left[\varphi_{J_1 M_1}, \varphi_{J_2 M_2}^\dagger \right] = \delta_{J_1 J_2} \delta_{M_1 M_2} . \quad (4.4)$$

The (normal ordered) energy-momentum tensor for the scalar field is

$$T^{ab} =: \frac{2}{3} \nabla^a \Phi \nabla^b \Phi - \frac{1}{3} \Phi \nabla^a \nabla^b \Phi - \frac{1}{6} g^{ab} ((\nabla \Phi)^2 + \frac{1}{6} \Phi^2) + \frac{1}{6} R^{ab} \Phi^2 : , \quad (4.5)$$

which is classically traceless. Furthermore, all three curvature invariants appearing in the trace anomaly, $C_{abcd}C^{abcd}$, $\square R$ and $G = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ vanish on $R \times S^3$, so that the trace of the energy-momentum tensor (4.5) also vanishes in the quantum theory:

$$T_a^a = 0 \quad \text{on } R \times S^3. \quad (4.6)$$

On S^3 with unit radius, $R_{0a} = 0$, $R_{ij} = 2\delta_{ij}$ and the energy density becomes

$$T^{00}|_{R \times S^3} =: \frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} \Phi \ddot{\Phi} + \frac{1}{12} \nabla^2(\Phi^2) : , \quad (4.7)$$

while

$$T^{0i}\Big|_{R \times S^3} =: -\dot{\Phi} \nabla^i \Phi + \frac{1}{6} \nabla^i \partial_t (\Phi^2) : . \quad (4.8)$$

Substituting the mode expansion (4.3) into (4.7) we form the moments,

$$\begin{aligned} \dot{T}_{JM}^{(+)}\Big|_{t=0} = & - \sum_{\substack{J_1 M_1 \\ J_2 M_2}} \mathcal{C}_{J_1 M_1 J_2 M_2}^{JM} \left\{ (J_1 + J_2 + 1) \left[(J_1 - J_2)^2 - \frac{1}{3} J(J+1) \right] \varphi_{J_1 M_1} \varphi_{J_2 M_2} \right. \\ & \left. + (J_1 - J_2) \left[(J_1 + J_2 + 1)^2 - \frac{1}{3} J(J+1) \right] \tilde{\varphi}_{J_2 M_2}^\dagger \varphi_{J_1 M_1} \Big|_{J_1 > J_2} \right\} \end{aligned} \quad (4.9)$$

as defined by (2.28). Here the notation

$$\tilde{\varphi}_{JM} \equiv \epsilon_M \varphi_{J-M} \quad (4.10)$$

has been used, and the symbol \mathcal{C} is an $O(4)$ angular momentum coupling coefficient defined by

$$\begin{aligned} \mathcal{C}_{J_1 M_1 J_2 M_2}^{JM} & \equiv \frac{\sqrt{V}}{\sqrt{(2J_1+1)(2J_2+1)}} \int_{S^3} d\Omega Y_{JM}^* Y_{J_1 M_1} Y_{J_2 M_2} \\ & = \frac{1}{\sqrt{(2J+1)}} C_{J_1 m_1 J_2 m_2}^{Jm} C_{J_1 m'_1 J_2 m'_2}^{Jm'} , \end{aligned} \quad (4.11)$$

where eq. (A.7) of the Appendix has been used and C denotes an ordinary $SU(2)$ Clebsch-Gordon coefficient. In the case of the lowest non-trivial moment $J = \frac{1}{2}$ we obtain the realization of the conformal generator $K_M^{(+)}$ in (2.27) for the conformal scalar field,

$$K_M^{(+)} = - \sum_{J M_1 M_2} \mathcal{C}_{J+\frac{1}{2} M_1 J M_2}^{\frac{1}{2} M} (J+1)(2J+1) \tilde{\varphi}_{J M_2}^\dagger \varphi_{(J+\frac{1}{2}) M_1} . \quad (4.12)$$

As discussed previously the moments $\dot{T}_{JM}^{(+)} + \dot{T}_{JM}^{(-)}$ generate the volume non-preserving spatial diffeomorphisms on S^3 . The volume preserving diffeomorphisms are generated by $X_{\mathcal{YM}^*} \equiv X_{JM}^{(+)} + X_{JM}^{(-)}$ with

$$X_{\mathcal{YM}^*}^{(+)}\Big|_{t=0} = -i \sum_{\substack{J_1 M_1 \\ J_2 M_2}} \mathcal{K}_{J_1 M_1 J_2 M_2}^{JM} \left\{ (J_1 - J_2) \varphi_{J_1 M_1} \varphi_{J_2 M_2} + (J_1 + J_2 + 1) \tilde{\varphi}_{J_2 M_2}^\dagger \varphi_{J_1 M_1} \right\}_{J_1 \geq J_2} \quad (4.13)$$

where the coupling coefficient with the transverse vector $O(4)$ spherical harmonic $\mathcal{Y}_{J\mathcal{M}}^i$ is defined by

$$\mathcal{K}_{J_1 M_1 J_2 M_2}^{J\mathcal{M}} \equiv \frac{1}{2\sqrt{(2J_1+1)(2J_2+1)}} \int_{S^3} d\Omega \mathcal{Y}_{J\mathcal{M}}^{i*} Y_{J_1 M_1} \nabla_i Y_{J_2 M_2} . \quad (4.14)$$

Relevant properties of the transverse vector harmonics on S^3 are given in the Appendix. The lowest moment of the volume preserving diffeomorphisms corresponding to $\xi^i = \rho^i$ of eq. (2.17) are precisely the 6 rotation generators of $SO(4)$ for the conformal scalar field,

$$R_{M_1 M_2} = \frac{i}{2} \sum_{J M_3 M_4} \int_{S^3} d\Omega \rho_{M_1 M_2}^i Y_{J M_4}^* \nabla_i Y_{J M_3} \varphi_{J M_4}^\dagger \varphi_{J M_3} . \quad (4.15)$$

Finally the Hamiltonian generator for the conformal scalar field on $R \times S^3$ is simply

$$H = \sum_{J M} (2J+1) \varphi_{J M}^\dagger \varphi_{J M} . \quad (4.16)$$

With the generators of the $SO(4,2)$ global conformal algebra for the scalar field (4.12), (4.15) and (4.16) in hand, one may verify that the quantum generators obey the algebra corresponding to the *classical* Lie brackets in (2.24), using the properties of the harmonics listed in the Appendix. There is *no* shift in the Hamiltonian for the matter field, in contrast to the -4 found in the *bc* ghost calculation of the previous section, since there is no analog of the ghost operator ordering question in the scalar sector.

We remark that the classical algebra of spatial diffeomorphisms as given by (2.5) is respected by the sum of positive and negative frequency generators constructed from the scalar field Fock operators above. However, taking commutators of the higher moments with *separate* positive and negative frequency parts of the diffeomorphism generators yields complicated expressions with no clear algebraic structure apparent, in contrast to the case in two spacetime dimensions where the full Virasoro algebra emerges this way. As mentioned in Section 2, whether or not such an extended algebra should exist, and if it does not what the consequences would be for the consistency of

reparametrization invariant quantum theories in four spacetime dimensions remains an interesting open problem.

In order to illustrate the method for constructing physical states which are invariant under infinitesimal coordinate diffeomorphisms, let us now apply the constraints to the general states of the matter field in the absence of gravitational field modes. The matter states are constructed in the usual way by operating with any number of creation operators φ_{JM}^\dagger on the Fock vacuum obeying

$$\varphi_{JM}|0\rangle = 0 \ , \quad (4.17)$$

on which H , $K_M^{(+)}$ and $R_{M_1 M_2}$ also vanish. In analogy with the canonical method in two dimensions we define the level of any eigenstate of the Hamiltonian by its integer eigenvalue,

$$H|N\rangle = N|N\rangle \ . \quad (4.18)$$

From the expression for the scalar field Hamiltonian (4.16) we observe that each creation operator φ_{JM}^\dagger contributes $2J+1$ units to the level of the state. Hence we may find the physical states of the scalar field by writing first the general linear combination of creation operators that create a given level N state, and then applying the diffeomorphism constraints to relate the coefficients of the linear combination of coefficients at this level, to see which states survive. This procedure is best carried out by beginning with the lowest levels and moving up in N one unit at a time.

At level one the only possible state is

$$|1\rangle = \varphi_{00}^\dagger |0\rangle \quad (4.19)$$

which survives imposition of all the spatial diffeomorphism constraints

$$\dot{T}_{JM}^{(+)}|1\rangle = X_{JM}^{(+)}|1\rangle = 0 \ . \quad (4.20)$$

At level two the general state is of the form

$$|2\rangle = \left(\alpha(\varphi_{00}^\dagger)^2 + \beta_M \varphi_{\frac{1}{2}M}^\dagger \right) |0\rangle . \quad (4.21)$$

Requiring that $\dot{T}_{\frac{1}{2}M}$ on this state vanishes yields $\beta_M = 0$, whereas the first term in the combination survives all the constraints for general α .

At level three the general state is

$$|3\rangle = \left(\alpha'(\varphi_{00}^\dagger)^3 + \beta'_M \varphi_{00}^\dagger \varphi_{\frac{1}{2}M}^\dagger + \gamma'_M \varphi_{1M}^\dagger \right) |0\rangle . \quad (4.22)$$

Now the application of the $\dot{T}_{\frac{1}{2}M}$ and \dot{T}_{1M} constraints require $\beta'_M = \gamma'_M = 0$ while the first term again survives all the constraints. Indeed applying the higher J constraint first eliminates the γ'_M term and then the β'_M term is eliminated for the same reason as at level 2.

Continuing in this way it is not difficult to convince oneself inductively that the only state at level N which survives all the spatial diffeomorphism constraints is

$$|N\rangle = (\varphi_{00}^\dagger)^N |0\rangle . \quad (4.23)$$

That this state does indeed satisfy all of the spatial constraints is easy to see directly from the forms of (4.9) and (4.13). For example, in the first term of $\dot{T}_{JM}^{(+)}$ the only non-trivial terms result from commuting the destruction operators with $J_1 = J_2 = 0$ through the creation operators in (4.23) above. But then the coupling coefficient \mathcal{C} is non-vanishing only for $J = 0$ which is multiplied by the coefficient $[(J_1 - J_2)^2 - \frac{1}{3}J(J+1)] = 0$. In the second term of $\dot{T}_{JM}^{(+)}$ for the same reason only $J_1 = 0$ can contribute, but then the condition $J_1 > J_2$ cannot be satisfied. Similarly neither term of the volume preserving diffeomorphisms (4.13) survives when operating on the state $|N\rangle$ above, which therefore satisfies the physical state conditions,

$$\dot{T}_{JM}^{(+)}|N\rangle = X_{JM}^{(+)}|N\rangle = 0 . \quad (4.24)$$

It is perhaps noteworthy to remark that even though we have not imposed the full time reparametrization constraints $\langle N'|T_{JM}|N\rangle = 0$ and indeed have argued in the previous sections that these constraints should not be imposed *a priori*, nevertheless we find that these conditions are satisfied as well for all J *except* $J = 0$ which is the Hamiltonian constraint and must be treated separately.

Finally we should apply the Hamiltonian constraint with the correct shift determined in the last section, namely $H|phys\rangle = 4|phys\rangle$. Clearly, of all the surviving states $|N\rangle$ in (4.23) this selects only the level $N = 4$ state as physical. The significance of this one surviving physical state is that it is created by the spacetime integral of the conformal weight 4 operator, $:\Phi^4:$ acting on the vacuum, $|0\rangle$. This integrated operator is the *only* non-trivial conformal scalar one can form in this free field example. Indeed under the conformal transformation from flat space to $R \times S^3$ given by (2.2), the volume element

$$d^4x = e^{-4\tau} d\tau d\Omega \rightarrow ie^{-4it} dt d\Omega \quad (4.25)$$

after analytic continuation to Lorentzian signature. Hence the integrated flat space operator $:\Phi^4:$ picks up a factor of e^{-4it} when transformed to $R \times S^3$ and we find

$$[H, \int dt d\Omega e^{-4it} : \Phi^4 :]|0\rangle = -i \int dt d\Omega e^{-4it} \partial_t : \Phi^4 : |0\rangle = 4 \int dt d\Omega e^{-4it} : \Phi^4 : |0\rangle \quad (4.26)$$

after integration by parts. Since the time integration in (4.26) selects only the time dependence e^{4it} from the normal ordered quantity $:\Phi^4:|0\rangle$, we see immediately from the mode expansion (4.3) that this yields precisely the physical state

$$|phys\rangle = (\varphi_{00}^\dagger)^4 |0\rangle . \quad (4.27)$$

Hence the integrated operator $\int d^4x : \Phi^4 :$ is an explicit realization in this free scalar theory of a weight 4 operator \mathcal{O}_4 creating a physical state in (3.1), as we discussed at the beginning of Section 3.

The fact that there is only one physical state (*i.e.* the “vacuum”) and only one non-trivial invariant operator surviving the diffeomorphism constraints in the scalar Φ theory is a reflection of the fact that there is no gravitational dynamics in this theory without a gravitational action. However, the correspondence of physical states with operators of dimension 4 is a general feature of any diffeomorphic invariant theory. The correspondence of scalar operators with physical states confirms the interpretation of the shift in the global Wheeler-DeWitt equation found in Sec. 3, and will be exploited further in the study of the effective theory of the conformal factor in paper II.

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Appendix

Because the symmetry group of S^3 is $O(4) \cong SU(2) \times SU(2)/Z_2$ the finite dimensional irreducible representations of this group are labeled by a pair of $SU(2)$ indices (J_1, J_2) each taking on integer or half-integer values. The scalar harmonics on S^3 , denoted here by Y_{JM} are eigenvectors of the scalar Laplacian on the sphere,

$$\begin{aligned} -\Delta Y_{JM} &= (2J)(2J+2)Y_{JM} = 4J(J+1)Y_{JM} , \\ M \equiv (m, m') &= (-J, -J), (-J+1, -J), \dots, (J, J-1), (J, J) \end{aligned} \quad (\text{A.1})$$

and belong to the (J, J) representation of $SU(2) \times SU(2)$ with multiplicity $(2J+1)^2$. They form a complete set in which to expand scalar functions regular on S^3 , are mutually orthogonal with respect to integration over $d\Omega$ and may be normalized in the standard way:

$$\int_{S^3} d\Omega Y_{J'M'}^* Y_{JM} = \delta_{J'J} \delta_{M'M} . \quad (\text{A.2})$$

Because S^3 is the group manifold of $SU(2)$, the spherical harmonics may also be viewed as rotation matrices that map the J representation of $SU(2)$ into itself. Hence the Y_{JM} are proportional to the $SU(2)$ Wigner D functions. In fact,

$$Y_{JM}(\alpha, \beta, \gamma) = \sqrt{\frac{2J+1}{V}} D_{mm'}^J(\alpha, \beta, \gamma) \quad (\text{A.3})$$

in the Euler angle (α, β, γ) parameterization of the three sphere,

$$d\Omega^2 = \frac{1}{4}(d\alpha^2 + d\beta^2 + d\gamma^2 + 2 \cos \beta d\alpha d\gamma) , \quad (\text{A.4})$$

in which α and γ have the range $[0, 2\pi]$ and $[0, 4\pi]$ (or *vice versa*), and β has the range $[0, \pi]$. In this parameterization the volume element on S^3 is

$$d\Omega = \frac{1}{8} d\alpha d\beta d\gamma \sin \beta \quad (\text{A.5})$$

and the volume $V = 2\pi^2$.

From the known properties of the Wigner D functions [8] we obtain the conjugation relation,

$$\begin{aligned} Y_{JM}^* &= \epsilon_M Y_{J-M} \quad \text{where} \\ \epsilon_M &\equiv (-)^{m-m'} \end{aligned} \quad (\text{A.6})$$

and the integral of three scalar harmonics,

$$\int_{S^3} d\Omega Y_{JM}^* Y_{J_1 M_1} Y_{J_2 M_2} = \sqrt{\frac{(2J_1+1)(2J_2+1)}{(2J+1)V}} C_{J_1 m_1 J_2 m_2}^{Jm} C_{J_1 m'_1 J_2 m'_2}^{Jm'}, \quad (\text{A.7})$$

in terms of two ordinary $SU(2)$ Clebsch-Gordon coefficients $C_{J_1 m_1 J_2 m_2}^{Jm}$, from which (4.11) of the text follows immediately.

The four scalar harmonics at $J = \frac{1}{2}$ lie in the $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2) \times SU(2)$ which is the vector representation of $O(4)$, and play a distinguished role in the analysis of the diffeomorphism constraints. Because these four harmonics span the space of translations of the S^3 in its embedding R^4 , the gradients $\omega_i^{(M)} = \nabla_i Y_{\frac{1}{2}M}$ satisfy the relations,

$$\begin{aligned} \nabla_i \nabla_j Y_{\frac{1}{2}M} &= -g_{ij} Y_{\frac{1}{2}M} \\ \Delta Y_{\frac{1}{2}M} &= \nabla_i \omega^{i(M)} = -3Y_{\frac{1}{2}M}, \end{aligned} \quad (\text{A.8})$$

which may be verified directly from the explicit representations (A.3) and (A.4) as well. Hence the four $\omega_i^{(M)}$ are the four proper conformal Killing vectors of S^3 ,

$$\nabla_i \omega_j^{(M)} + \nabla_j \omega_i^{(M)} = \frac{2}{3} g_{ij} \nabla_k \omega^{k(M)}, \quad (\text{A.9})$$

and generate four special volume non-preserving coordinate transformations of the sphere. From this relation and the definition (2.20) one immediately derives (2.21) of the text and the fact that $\kappa_M^{(\pm)}$ satisfy the conformal Killing equation of the full metric on $R \times S^3$. Furthermore, using (A.8) we observe that

$$\nabla_j \left(\nabla_i Y_{\frac{1}{2}M'}^* \nabla^i Y_{\frac{1}{2}M} \right) = \nabla_j \left(-Y_{\frac{1}{2}M'}^* Y_{\frac{1}{2}M} \right) \quad (\text{A.10})$$

so that the quantities in parentheses are equal up to a constant. The constant may be evaluated by taking the volume integral of each on S^3 . In this way, we find

$$(\nabla_i Y_{\frac{1}{2}M'}^*)(\nabla^i Y_{\frac{1}{2}M}) + Y_{\frac{1}{2}M'}^* Y_{\frac{1}{2}M} = \frac{4}{V} \delta_{M_1 M_2} \delta_{M'_1 M'_2} . \quad (\text{A.11})$$

The six Killing vectors ρ^i of the sphere may also be expressed in terms of these same $J = \frac{1}{2}$ scalar harmonics by eq. (2.17) of the text. Using this definition, the definition of the Lie bracket contained in eqs. (2.5) and (2.6) and the relation (A.11), it is straightforward to derive the classical Lie algebra (2.24) of the conformal group of the Einstein universe $R \times S^3$.

The six Killing vectors ρ^i are actually the first non-trivial representatives of the transverse vector harmonic functions on the 3-sphere, which we denote generally by $\mathcal{Y}_{J\mathcal{M}}^i$. These belong to the $(J + \frac{1}{2}, J - \frac{1}{2})$ or $(J - \frac{1}{2}, J + \frac{1}{2})$ representations of $SU(2) \times SU(2)$, so that the magnetic index \mathcal{M} ranges over $2 \times (2J + 2)(2J) = 8J(J + 1)$ values and $J \geq \frac{1}{2}$. The vector harmonics are transverse,

$$\nabla_i \mathcal{Y}_{J\mathcal{M}}^i = 0 \quad (\text{A.12})$$

and are eigenfunctions of the Laplacian on S^3 ,

$$-\Delta \mathcal{Y}_{J\mathcal{M}}^i = [4J(J + 1) - 1] \mathcal{Y}_{J\mathcal{M}}^i . \quad (\text{A.13})$$

At $J = \frac{1}{2}$ the $(1, 0)$ and $(0, 1)$ representations give $3 + 3 = 6$ transverse vector harmonics which are precisely the 6 Killing vectors of S^3 ,

$$\nabla_i \rho_j + \nabla_j \rho_i = 0 , \quad (\text{A.14})$$

and take the special form (2.17). In this case the index \mathcal{M} takes on the six values corresponding to the adjoint representation of $O(4)$, labeled by an anti-symmetrized pair of four-vector indices (M_1, M_2) . None of the higher $J > \frac{1}{2}$ vector harmonics can be

written as simply in terms of products of scalar harmonics and their gradients. Taking the divergence of the Killing equation and using $[\nabla_j, \nabla_i]\rho^j = R_{ji}\rho^j = 2\rho^i$ on the unit sphere gives

$$\Delta\rho_i + 2\rho_i = 0 \quad (\text{A.15})$$

which agrees with the eigenvalue equation (A.13) for the $J = \frac{1}{2}$ vector harmonics [8].

All of the vector harmonics $\mathcal{Y}_{J\mathcal{M}}^i$ are mutually orthogonal and may be normalized in the standard way, *i.e.*

$$\int_{S^3} d\Omega \mathcal{Y}_{J'\mathcal{M}'}^{i*} \mathcal{Y}_{J\mathcal{M}}^i = \delta_{J'J} \delta_{\mathcal{M}'\mathcal{M}} . \quad (\text{A.16})$$

Together they form a complete basis for the volume preserving diffeomorphisms of the sphere. The Killing vectors defined by (2.17) are normalized differently, in order to satisfy the classical Lie algebra (2.24) and obey

$$\int_{S^3} d\Omega \rho_{M_1 M_2}^{i*} \rho_{i M_1 M_2} = \frac{V}{2} (\delta_{M_1 M_3} \delta_{M_2 M_4} - \epsilon_{M_1} \epsilon_{M_2} \delta_{M_1 - M_4} \delta_{M_2 - M_3}) \quad (\text{A.17})$$

instead of (A.16). Finally, in deriving the quantum realization of the classical conformal algebra (2.24) one also encounters

$$\begin{aligned} \int_{S^3} d\Omega \rho_{M_1 M_2}^i \rho_{M_3 M_4}^{j*} \nabla_i \rho_{j M_5 M_6} = \\ \frac{iV}{4} \left\{ \delta_{M_4 M_6} (\delta_{M_1 M_3} \delta_{M_2 M_5} - \epsilon_{M_1} \epsilon_{M_2} \delta_{-M_1 M_5} \delta_{-M_2 M_3}) \right. \\ + \epsilon_{M_1} \epsilon_{M_3} \delta_{-M_3 M_6} (\delta_{-M_1 M_5} \delta_{M_2 M_4} - \delta_{-M_1 M_4} \delta_{M_2 M_5}) \\ - \delta_{M_4 M_5} \epsilon_{M_5} \epsilon_{M_6} (\delta_{M_1 M_3} \delta_{M_2 M_6} - \epsilon_{M_1} \epsilon_{M_2} \delta_{-M_1 M_6} \delta_{-M_2 M_3}) \\ \left. - \epsilon_{M_1} \epsilon_{M_3} \epsilon_{M_5} \epsilon_{M_6} \delta_{-M_3 M_5} (\delta_{-M_1 M_6} \delta_{M_2 M_4} - \delta_{-M_1 M_4} \delta_{M_2 M_6}) \right\} , \end{aligned} \quad (\text{A.18})$$

which is evaluated by repeated use of the relations (A.8) and (A.11) above.